The Mathematics of NIELS HENRIK ABEL.
Continuation and New Approaches in Mathematics During the 1820s

PhD dissertation

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