The Mathematics of NIELS HENRIK ABEL.
Continuation and New Approaches in Mathematics During the 1820s

PhD dissertation

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Summary

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The present PhD dissertation uses the mathematics of the Norwegian NIELS HENRIK ABEL (1802–1829) as a framework for describing and analyzing trends in the development of mathematics during the first half of the 19th century. ABEL's mathematics is read and interpreted in its context and used to describe a fundamental change in mathematics in the early 19th century in which concepts replaced formulae as the basic objects of mathematics.

The dissertation is structured into five parts: 1) an introductory part consisting of biographical and other historical framework, 2) three descriptive parts each devoted to a particular theme analyzed from a particular discipline in ABEL's mathematical production, and 3) a part comprising the syntheses of a general transition in mathematics in the early 19th century as seen from the perspective of ABEL's works.

Introduction. In the introductory part, ABEL's biography is described to point out some of the formative instances in the creation of one of the important mathematicians of the first half of the 19th century. Because ABEL's biography has been written repeatedly — and recently in an excellent cultural biography — the biography is only intended to locate ABEL's production in its contexts of his life and the mathematics of his time.

New questions: Algebraic solvability. The first of the three studies of ABEL's mathematics deals with his contributions to the theory of equations. It is illustrated how ABEL was led to ask a new kind of question of the solvability of equations which would have seemed both counter intuitive and futile to mathematicians a few generations before. With the foundation in works of LAGRANGE and CAUCHY, ABEL was able to prove that the algebraic solution of general quintic equations was impossible. This result restricted the class of solvable equations and separated it from the class of

all polynomial equations. In another line of research, ABEL proved that an extensive class of equations — later called *Abelian equations* — were algebraically solvable. Compared with the previous result, the solvability of the *Abelian* equations showed that the extension of the concept of solvable equations did not collapse. Subsequently, this branch of the theory of equations became a question of delineating the extension of solvable equations, i.e. of drawing the border between solvable and unsolvable equations by some other criterion. ABEL commenced research on this issue but had to leave it incomplete. In part **??**, ABEL's research on all these issues is carefully analyzed based on the works of his main predecessors and contemporaries. The reception of ABEL's research and the subsequent development of the theory is also addressed.

New epistemic standards: Rigorization. ABEL's devotion to and adaption of the Cauchian rigorization movement is the topic of part **??**. By describing ABEL's critical attitude towards the existing practices of rigor and his publications on the binomial theorem and a certain type of criteria of convergence, it is illustrated how the new epistemic standards were manifesting themselves in a period of rapid transition in analysis. Starting with a description of the Eulerian focus on algebraic equality, it is described how CAUCHY's new emphasis on arithmetical equality effected the central concepts of continuous functions and converget series. Furthermore, the so-called *exception* which ABEL presented to a theorem by CAUCHY is treated in some detail because it will later prove to be an important example in the description of the change from formula based to concept based mathematics.

New objects: Elliptic and higher transcendentals. The third and final pillar of ABEL's mathematics which is treated in the present dissertation concerns his works on elliptic and higher transcendentals. Selected aspects of his work are again presented and discussed from a diachronical viewpoint. In this case, special emphasis is given to the way ABEL was led to his formal inversion of elliptic integrals into elliptic functions. Furthermore, ABEL's means of obtaining workable representations of the formally defined object is described. Thereafter, special attention is paid to the techniques which he employed in studying transcendentals and it is illustrated how *algebraic* methods figured prominently in his toolbox. In the process, it is also described how his style of argument often relied essentially on manipulations of formulae in ways which could sometimes lead to results which were true "in general". Finally, the changing internal relationships between definitions and results are illustrated by describing various ways towards a general theory of elliptic functions.

Syntheses. In the ultimate part, the preceding descriptions and discussions of ABEL's mathematics are thrown into perspective by arguing that the development of mathematics in the early 19^{th} century can be understood as a change of paradigms: An Eulerian, formula based paradigm is contrasted with a concept based paradigm. Various aspects of ABEL's works — including delineation problems, ABEL's *exception*, and the nature of arguments which are only true "in general" — are then all interpreted based on this transition in paradigms.