

Chapter 21.

ABEL's mathematics and the rise of concepts

An extract from

The mathematics of NIELS HENRIK ABEL.
Continuation and New Approaches in
Mathematics During the 1820s.

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CHAPTER 21

ABEL's mathematics and the rise of concepts

In the preceding parts, I have presented and discussed the main parts of ABEL's mathematical production ranging from the theory of equations over the installation of rigor in the theory of series to the exploding field of elliptic and higher transcendentals. In addition, in each part, I have simultaneously addressed three broader themes: the rise of new questions with new kinds of answers, the change in the standards of doing mathematics, and a change in the objects and methods of mathematics.

In this concluding part, I unify these themes by arguing that they are signs of a rise of *concept based mathematics*. In four steps, it will be argued that a large part of the development in mathematics in the early 19th century can appropriately be analyzed from the perspective of a change in the objects with which mathematics dealt. Firstly, the change is introduced by defining and discussing the "paradigms" of *formula based mathematics* and *concept based mathematics*. Secondly, these concepts are employed to present a brief analysis of how the ways of introducing objects into mathematics developed in the early 19th century. Thirdly, the changing role of counter examples in the two paradigms is discussed in some details. Finally, some reservations to the analysis are presented before a conclusion is drawn. Due to the limited scope of ABEL's mathematical production, my frame of interpretation can only be preliminary and I hope to develop it further through subsequent research by involving the works of other mathematicians.

21.1 From formulae to concepts

It seems fair to state that in the early 19th century, mathematics changed quite dramatically.¹ Any comparative and contextualized reading of the works of, say, EULER and WEIERSTRASS or RIEMANN will reveal that the problems, methods, and styles of analytical mathematics developed immensely and changed fundamentally during the century from the 1750s to the 1850s. In my view, a large part of the change in mathematics in the period can be understood by analyzing a fundamental change in the basic objects of the combined discipline of algebra and analysis.

My description of changes in mathematics will — dictated by the scope of the previous parts — mostly deal with the development of the algebraic and analytic disciplines. In section 21.4, the applicability of the analyses outside these disciplines is briefly addressed.

It will be argued that mathematics in the 18th century was tied to *formulae* and that mathematicians worked within a framework which was — in essential ways — adapted to these objects. In the early 19th century, so it is argued, these basic objects were gradually replaced by *concepts* and the change was so fundamental that it influenced all layers of mathematical knowledge and knowledge production.

To allow for a more precise discussion, tentative definitions of the two styles (paradigms) of mathematics are given below. I have adopted the excessively broad *Kuhnian* term “paradigm” to include the entire mental horizon of the group of mathematicians who worked in the tradition. At the same time, I have introduced catch-word characterizations of the paradigms by terming them *formula based* and *concept based*. Below, the paradigms and their relations will be discussed further and certain relevant aspects of the preceding presentation of ABEL’s mathematics will be analyzed.

21.1.1 The Eulerian paradigm of *formula based mathematics*

By *formula based mathematics*, I mean to indicate a paradigm prevalent in the 18th century in which formulae were the carriers of mathematical knowledge. Formulae were both the results and the methods of mathematics, and mathematicians thought *about* and *in terms of* formulae. Mathematical results were derived through strings of explicit, formal manipulations of representations (formulae) and were stated in terms of new formulae.

The essential notion of *formula* can be thought of as representations of mathematical objects by symbols. However, such interpretations tend to be anachro-

¹Whether or not the early 19th century is an apt periodization in the history of mathematics has been discussed, though. See e.g. (Mikulinsky 1982; Otte 1982).

nistic and beside the point because — as I shall argue — the formulae were the *basic* objects of mathematics and only gradually became representations of other objects.²

In analysis, the primary occurrence of formulae was in the form of functions; the study of functions had been based on the study of their algebraic formulae. For these reasons, this paradigm could also have been named *function based mathematics* if it was only to apply in analysis. However, in the algebraic discipline such a description would be misguided precisely because not functions but formulae were at the centre of mathematical reasoning (see e.g. section ??). If any other name should be used for formula-based mathematics, the term *Eulerian paradigm* might be well suited.

For the present purpose, formula based mathematics is best thought of in terms of e.g. EULER's introduction and manipulation of various algebraic expressions in analysis. ABEL's mathematics also frequently exhibits key characteristics of this paradigm, e.g. in his manipulations of formulae in the *Recherches* or the latter part of the binomial paper (see chapter ?? and ??, respectively). On both these occasions, ABEL based his deductions on sequences of step-wise manipulations of formulae to obtain results which were, themselves, formulae.

21.1.2 A new paradigm of *concept based mathematics*

The anti-thesis to formula based mathematics in the present context is termed *concept based mathematics*. In analogy with the formula based version, this paradigm emphasized thought in and about concepts by which I mean classes of objects. Concept based mathematics deals primarily with defining, representing, and relating concepts. The collection of objects which fall under a concept is called the *extension* or *domain* of the concept.

Typically, concept based mathematics could be concerned with e.g. *continuous functions*, *differentiable functions*, or *algebraically solvable equations*. The mathematical theorems dealing with concepts would then contain results relating these, e.g. by pointing out their differences or their overlaps or by relating one concept to another. In a truly concept based approach to mathematics, even representations become theorems relating concepts; ABEL's deduction of the normal form for (explicit) algebraic expressions stated that the two concepts were identical (see figure 21.1, section ??, and below).

For concept based mathematics to be efficient, specific knowledge of the individual objects within a concept has to fade in importance. Individual objects would serve important roles as examples and counter examples but the defini-

²In the 17th and part of the 18th century, formulae had also been representations of e.g. curves (see section ??). However, as described, they became the primary objects in EULER's new version of analysis.

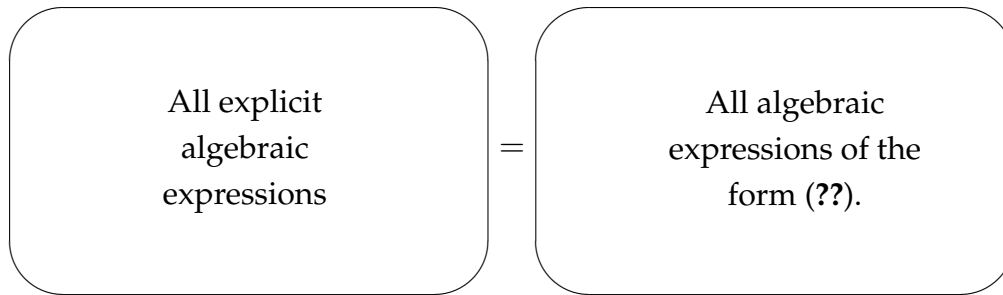


Figure 21.1: The equality of the concepts of explicit algebraic expressions and ABEL’s normal form.

tions of the concepts must possess qualities which make them useful and central in the investigation of the concept. Such investigations, in turn, can benefit from the shift of focus onto concepts and produce results which were impossible (or very difficult) if only individual objects were considered. Thus, in order to analyze concept based mathematics, the role of definitions, representations, and arguments of relation and delineation of concepts become key points of enquiry.

21.1.3 The shift from formula based to concept based mathematics

The purpose of introducing the paradigms of formula based and concept based mathematics is to characterize the development in the early 19th century as a transition from the former to the latter. This transition manifests itself in various ways which interact with the changing basic objects of mathematics. The questions asked, the tools employed to answer these questions, and the types of answers which are possible and expected all change as consequences of this shift.

In the first half of the 19th century, some mathematicians were aware that their style of mathematics differed essentially from the standards of their time. ABEL expressed how heavy loads of computations could hamper the progress of research,³ and GALOIS described his own works as “analysis of analysis” which would reduce the hitherto dominating calculations to particular cases.⁴ This awareness of the transition grew stronger during the century and towards the end of the 19th century, mathematicians became increasingly explicit about it. For instance, FERDINAND RUDIO (1856–1929) wrote:

“The essential principle of the newer mathematical school, which is established by Gauss, Jacobi, and Dirichlet, is that whereas the older one sought to reach the goal by lengthy and complicated calculations

³(Abel 1839 [1828], 217–218).

⁴(Galois 1831, 11).

(as even still in Gauss' *Disquisitiones*) and deductions — it comprises an entire field by avoiding those and applying a genius method in a main idea and simultaneously presents the end result in its highest elegance by a single strike. While the former [the older approach] after a long sequence eventually reached a fertile ground by progressing from theorem to theorem, the latter [the new approach] immediately produces a formula in which the complete sphere of truths of an entire field is compactly contained and only ought to be extracted and expressed. In the old way, one could also — if need be — prove theorems; but only now can the true nature of the entire theory be seen, its internal gears and wheels."⁵

RUDIO clearly expressed the transition; the *formula* which he describes as the product of the new approach is not a formula in the present sense but rather a *theorem*. A number of similar statements can be found by other late-19th century mathematicians, e.g. CHRISTIAN FELIX KLEIN (1849–1925) who described how DIRICHLET would avoid long computations in favor of acute logical analyses.⁶

This shift in the roles of formulae and concepts has been noticed and investigated from slightly different perspectives by historians of mathematics. In particular, it has been addressed by HANS NIELS JAHNKE and by DETLEF LAUGWITZ, who pointed to the significant influence which RIEMANN had in bringing about the eventual change of paradigms.⁷

The basic objects of mathematics. The definitions of the paradigms suggest that the purported shift from formula based to concept based mathematics was a question of the size of the domains of mathematical results. Interpreted purely as a change in domains, the new approach could be seen as consisting of results which are simultaneously true for a number of objects of the old paradigm (formulae). However, there is more to the transition than this; it concerns a real and

⁵"Das wesentliche Princip der neueren mathematische Schule, die durch Gauss, Jacobi und Dirichlet begründet ist, ist im Gegensatz mit der älteren, dass während jene ältere durch langwierige und verwickelte Rechnung (wie selbst noch in Gauss' *Disquisitiones*) und Deduktionen zum Zweck zu gelangen suchte, diese mit Vermeidung derselben durch Anwendung eines genialen Mittels in einer Hauptidee die Gesamtheit eines ganzen Gebietes umfasst und gleichsam durch einen einzigen Schlag das Endresultat in der höchsten Eleganz darstellt. Während jene, von Satz zu Satz fortschreitend, nach einer langen Reihe endlich zu einigem fruchtbaren Boden gelangt, stellt diese gleich von vorn herein eine Formel hin, in welcher der vollständige Kreis der Wahrheiten eines ganzen Gebietes konzentriert enthalten ist und nur herausgelesen und ausgesprochen zu werden darf. Auf die frühere Art konnte man die Sätze zwar auch zur Not beweisen, aber jetzt sieht man erst das wahre Wesen der ganzen Theorie, das eigentliche innere Getriebe und Räderwerk." (Rudio 1895, 894–895)

⁶(Klein 1967, 250).

⁷See e.g. (Jahnke 1987) and (Laugwitz 1999, 293–340). These are both very interesting works dealing with discussions similar to the present one but from slightly different perspectives.

fundamental change from formulae to concepts as the basic objects of mathematics. In the one extreme, a manipulation of a particular algebraic formula might produce another algebraic formula which would then be a mathematical result. At the other end of the spectrum, a number of results developed in the 19th century pointed out the differences between important concepts such as continuous and differentiable functions or proved that particular classes of functions could be represented in particular ways. The ability to state and prove results for abstractly defined classes of objects is one of the main aspects of the rise of concept based mathematics. Similarly, the issues of *relating concepts* and *representing concepts* are two of the central topics in a fully fledged concept-based version of mathematics.

The techniques and questions of mathematics. Connected to the transition in the basic objects of mathematics, the techniques and questions of mathematics also underwent fundamental changes. The types of questions asked and the methods for answering them were not the same in the two paradigms. In the formula based paradigm, mathematical texts could be made up of long sequences of manipulations which transformed one formula into others or answered particular questions by developing formulae which “solved” them. At times, concept based mathematics could apply the manipulation of “representations” provided that some representation result made it relevant. But more typically and interestingly, in concept based mathematics, statements about the extension of a concept grew to become the key results of mathematics. A particularly illustrative example of these questions has been presented in part ?? where ABEL’s research on the quintic first showed that not all polynomial equations were solvable, i.e. the concepts of *polynomial equation* and *algebraically solvable equations* were distinct, although related. Later in his research, ABEL’s proof of the algebraic solvability of *Abelian equations* was another almost prototypical concept based result, at least in its final formulation. This result showed that the concept of *Abelian equations* was contained in the concept of *algebraically solvable equations*. When he first encountered *Abelian equations* in connection with the division problem (see section ??), ABEL’s argument relied extensively on his particular knowledge of the individual objects and was thus much more formula based.

The styles of mathematics. Not surprisingly, the changing techniques of mathematics manifested themselves at the textual level. Because formulae had been the carriers of knowledge and argument in the formula based paradigm, mathematical publications relied extensively on the powers of formulae and mathematical texts could be dominated by strings of explicit manipulations of formulae. Eventually, a conclusion could be stated in the form of a theorem. In the concept based paradigm, a Euclidean style with its emphasis on definitions, theorems,

and proofs became the customary style of written mathematics. This presentational style emphasized the precise statement of assumptions and the internal relations between concepts and theorems.

A revolution? When a change of paradigms is involved, the question of *revolutions* naturally arises. According to the recent debate, revolutions in mathematics appears not to be the most apt scheme of interpreting the history of the discipline.⁸ In particular, the requirements of incommensurability seems to prohibit revolutions appearing in mathematics because the truth status of mathematical statements apparently never changes. This also seems to apply to the change of paradigms discussed here. During and after the transitional period, mathematicians devoted an effort to reconstructing and re-interpreting the established knowledge to make it fit into the new system. This is particularly visible in analysis where CAUCHY's deliberate redefinition of basic notions and priorities changed the status of certain results and perceptions. As a result, men like ABEL sought to refound the theory in such a way that absolute truth was retained by making explicit the domains of validity for the statements. This process can be called *critical revision* and its general success precludes revolutions in mathematics. In section 21.3, the role of counter examples in the early 19th century is invoked to shed some light on this discussion.

21.2 Concepts and classes enter mathematics

As the basic objects of mathematics went from formulae to concepts, new methods and standards for introducing the objects were developed and the internal purpose of mathematical research also changed.

21.2.1 Defining concepts

While an object in the formula based paradigm could be introduced by merely exhibiting its formula, the introduction of concepts into the concept based paradigm required more sophisticated methods. However, these methods were not necessarily new — a number of them had been around since the births of the Euclidean style of mathematics and Aristotelian logic in Ancient Greece. In the present context, two aspects of the new importance given to definitions deserve special attention. First, genetic definitions and nominal definitions are discussed and their interactions described. Second, the introduction of concepts with special properties through careful definitions is emphasized.

⁸See primarily the articles in (Gillies 1992).

Genetic and nominal definitions. Concepts were often introduced by either genetic or nominal definitions. A genetic definition consists of prescribing the way the concept is constructed from other, simpler concepts whereas a nominal definition simply associates a name to something. Typical examples of a genetic definitions in the present material include EULER's definition of functions and ABEL's definition of explicit algebraic expressions (see sections ?? and ??, respectively). These two examples also illustrate a very important difference in defining concepts: EULER's definition was purely nominal whereas ABEL put his definition to essential use in obtaining his normal form of explicit algebraic expressions.⁹ Nominal definitions were being discussed in the early 19th century but the debate mainly centered on the ancient question whether or not nominal definitions implied the existence of any objects under the concept being defined.¹⁰ The main objection against nominal definitions from a concept based paradigm could have been that they were not useful in obtaining knowledge of the concept being defined.¹¹

Definition by desired property. The ultimate way of associating knowledge through definitions would be to let properties serve as definitions. In a sense, this is the final lesson of LAKATOS' *Proofs and Refutations*: the polyhedra which satisfy the Eulerian formula are collected as a concept and called Eulerian polyhedra and of those, the Eulerian formula is trivially true.¹² Nevertheless, such definitions can be extremely useful in order to investigate other properties. With CAUCHY's fundamental shift towards arithmetic — rather than algebraic — equality, the numerical convergence of partial sums of series was given prime importance by using it to define *convergent series*.¹³ Thus, a property of formal series — which could be numerically convergent or not — was used to define a concept which was subsequently promoted and investigated. A similar change went on in the theory of elliptic functions where ABEL's original formal inversion of elliptic integrals was replaced by other definitions of elliptic functions. Many of the definitions of elliptic functions following ABEL's original one turned properties — which were *results* in ABEL's theory — into definitions. The motivations for this change in the status of properties of elliptic functions are many; rigor and theoretical applicability figure prominently among them.

⁹See e.g. (Laugwitz 1999, 311) and section ??.

¹⁰Among the mathematicians involved were GERGONNE and OLIVIER. See e.g. (Otero 1997, 74–81) and (Olivier 1826).

¹¹In (Grabiner 1981b), GRABINER has similarly emphasized the role which CAUCHY's new definitions played for his foundation for the calculus.

¹²(Lakatos 1976).

¹³See section ??.

21.2.2 Relating concepts

As a result of the transition, theorems about concepts and relating concepts came to dominate mathematics. Two types of relations among concepts were of principal importance: the representation of concepts and the determination of the extension of concepts.

Representing concepts. Mathematical symbolism and formulae had proved to be an extremely useful and powerful tool in developing theories in the formula based paradigm. In order to be able to continue this line of research into the concept based paradigm, representations of concepts became quite important. Central instances include ABEL's classification of explicit algebraic expressions and the multitude of representations of elliptic functions which he developed. A particularly revealing example of the benefits of representations was illustrated in section ?? where ABEL's use of infinite representations in the theory of transformation was discussed. The study of concepts in their entirety and not the individual objects meant that statements concerning the impossibility of certain representations could also be made and proved. This is particularly true of ABEL's proof of the unsolvability of the quintic (see chapter ??) in which a representation of all explicit algebraic expressions was proved not to be sufficiently powerful to encompass the implicitly defined algebraic expression corresponding to a solution of the general fifth degree equation. The very same example also serves to illustrate the problem of distinguishing concepts.

Distinguishing concepts. With the focus on concepts, it also became an important question to determine whether two concepts were identical or differed in their extensions. One of the very best examples is the debate which during the 19th century separated the concepts of *continuous* and *differentiable* functions by constructing ever more pathological functions belonging to the former concept but not to the latter one.¹⁴ The process of investigating concepts can often be thought of as a dialectic effort alternating between limiting and extending the domain of the concept. In pointing out the existence of objects within a concept and differences between concepts, examples and counter examples became very important mathematical tools. A number of similar uses of examples and counter examples can also be found in ABEL's works. The most conspicuous example is in the theory of equations where ABEL's proof on the quintic interpreted as limiting the class of solvable equations is precisely in this line of results. The use of counter examples as limitations on concepts is a quite modern one which is only meaningful within the concept based paradigm (see section 21.3).

¹⁴See (Volkert 1987; Volkert 1989).

Delineating concepts. One type of questions concerning the relation between concepts is so important that it deserves special attention; I have called it *delineation of concepts*. This notion refers to a set of questions which concern the precise characterization of the extension of a concept by some external and applicable criterion. In other words, these questions ask for a (feasible) method of determining whether a given particular object falls within the extension of a concept or not. In analogy with the steps of limiting and extending the extension of concepts (figures ?? and ??, respectively), a graphical representation of the delineation of concepts is produced in figure 21.2.

ABEL's unfinished research on a general theory of algebraic solvability was motivated by precisely this problem of determining whether or not a given equation could be solved algebraically. Similarly, the search for complete criteria of convergence also sought to delineate the extension of convergent series once and for all. The search for delineation of solvable equations came to a fruitful conclusion when GALOIS' criterion was finally accepted as an answer. The complete determination of the concept of convergent series was never so successful; the only complete characterization obtained was the *Cauchy criterion* (see page ??) which did not fully meet the demand for being external and easy to apply.

21.3 The role of counter examples

It has been described how the problem of investigating the extension of concepts led to a particular use of examples and counter examples. Inspired by ABEL's curious remarks about his "exception" to *Cauchy's theorem* (see section ??), I suggest that counter examples played fundamentally different roles in the two paradigms at stake.

21.3.1 Theorems with exceptions

In his binomial paper, ABEL described how he found *Cauchy's theorem* to "suffer exceptions" and I find it puzzling to investigate how theorems could possibly admit exceptions in the 1820s. First, however, the very phrasing of ABEL's statement must be considered. Then, by way of recalling arguments carried out "in general", the connection between exceptions and the formula based paradigm opens up.

The authenticity of the wording. One may try to explain ABEL's wording away as a result of his shyness and veneration for CAUCHY. For instance, in his criticism of OLIVIER, ABEL used the mild phrase "this part does not seem to be true" in the printed version rather than the more severe judgement "Mr. Olivier is

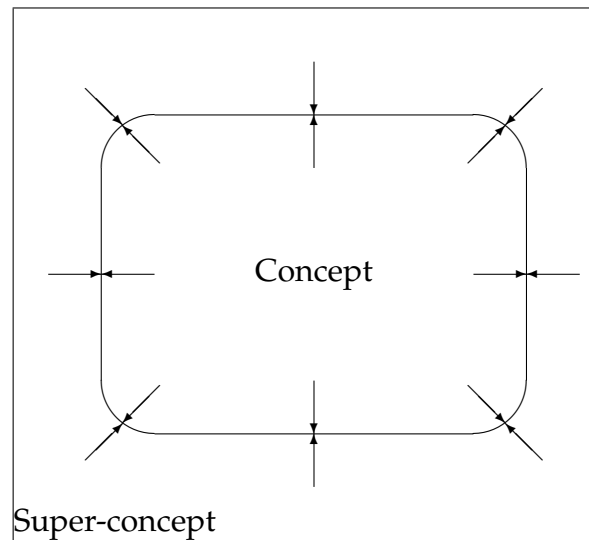


Figure 21.2: Delineating the border between a concept and its super-concept.

seriously mistaken” which we find in ABEL’s notebooks.¹⁵ This would suggest that exceptions were a milder form of criticism than outright counter examples or even paradoxes which were also terms found in ABEL’s vocabulary. Besides, the problem remains that we only have CRELLE’s translation of ABEL’s original manuscript at our disposal and single words in an edited manuscript can easily be over-interpreted. Nevertheless, when CAUCHY eventually reacted to ABEL’s exception, he did so explicitly stating that he wanted to correct the statement of his theorem so that “it no longer admitted exceptions” (see section ??).¹⁶ Thus, the word “exceptions” was chosen in this connection, and I believe that the following interpretation makes it plausible that ABEL actually meant that *Cauchy’s theorem* suffered an exception — or rather, a number of exceptions.

Arguments carried out “in general”. In the formula based paradigm, a situation sometimes arose in which the formula carrying the mathematical argument did not apply in all (numerical) cases. A number of such examples have been described above starting with EULER’s awareness that peculiar numerical results could emerge if specific values were inserted in expressions which were formally equal.¹⁷

As GRABINER describes,¹⁸ LAGRANGE held a strong and lifelong belief in the concept of “the general” in mathematics. Not only could formulae which were valid “in general” be of high importance — a general approach and system in mathematics was also strived for. When LAGRANGE presented his argument that

¹⁵See section ??.

¹⁶“Au reste, il est facile de voir comment on doit modifier l’énoncé du théorème, pour qu’il n’y ait plus lieu à aucune exception.” (Cauchy 1853, 31–32)

¹⁷See section ??.

¹⁸See (Grabiner 1981a, 317) or (Grabiner 1981b, 39).

“all” functions could be expanded in *Taylor series*, he was also aware that this might indeed fail to be true for particular functions at particular points.¹⁹ However, these instances where the general results failed to be true were particular, peculiar, and of little interest to mathematicians ascribing to the formula based paradigm.

In connection with ABEL’s *Paris mémoire*, an even more elaborate case was presented. At a crucial point in his argument to determine the number μ of independent integrals, ABEL employed a generalized degree operator called h .²⁰ Just as is the case for the ordinary degree operator of polynomials $\deg P$ (which ABEL also used), the degree of a sum may fail to be the maximum of the two degrees,

$$\deg (P_1 + P_2) \stackrel{?}{=} \max \{ \deg P_1, \deg P_2 \}, \quad (21.1)$$

if $\deg P_1 = \deg P_2$. However, in the *Paris mémoire*, ABEL was not interested in peculiarities and he simply argued that the equality corresponding to (21.1) was true “in general”, i.e. with the exception of some particular cases of little interest (see page ??). Once the paradigms had shifted, the precise determination of the number μ (called the genus) and the investigation and exposure of the necessary assumptions became a hot topic of mathematics.

In a similar situation, ABEL concluded his summary of well known properties of elliptic functions in the *Précis* by the statement:

“The formulae which have been presented above uphold with certain restrictions if the modulus c is arbitrary, real or imaginary.”²¹

ABEL’s way of obtaining the important formulae — often through tedious manipulations of infinite representations — could result in particular cases for which the formulae degenerated or produced false results. However, these cases were few and did not constitute an obstacle to presenting the formulae.

The number of exceptions. As indicated, in the formula based paradigm, results which suffered a few exceptions could still be very useful and the existence of exceptions did not immediately lead to the overthrow of theorems. This suggests an interesting way of interpreting the last part of ABEL’s famous footnote: Besides introducing his exception, ABEL also claimed that many similar functions existed. This indicates that the number of exceptions played a role. A similar remark can also be found in connection with CAUCHY’s example of a non-zero

¹⁹See e.g. (Lagrange 1813, 29–30).

²⁰See section ??.

²¹“Les formules présentées dans ce qui précède ont lieu avec quelques restrictions, si le module c est quelconque, réel ou imaginaire.” (Abel 1829, 245)

function whose *Maclaurin series* is the zero-function,²²

$$f(x) = e^{-\frac{1}{x^2}}.$$

This function represented an exception to the general belief in the expansion in power series which laid at the heart of the Lagrangian approach to analysis.²³ In 1822 and 1829,²⁴ CAUCHY presented this example and observed how to construct other functions with the same property of not being represented by their *Maclaurin series* except at a single point.

Both these examples suggest that if theorems in the formula based paradigm contained a quantification as “for all ...”, it might be necessary to introduce a statistical interpretation of the for-all quantification as KLAUS VOLKERT has suggested.²⁵ Exceptions and their numbers were noticed but no clear distinction between refuted (false) theorems and theorems with exceptions can be drawn. Theorems could be valid even if they suffered exceptions as long as the known exceptions were not too many or too important.

Exceptions and the formula based paradigm. Thus, the argument is that exceptions did have a place in mathematics of the formula based paradigm. The highly computational deductions based on long sequences of manipulations with finite and infinite representations occasionally led to results which were (only) true “in general”. Instead of discarding such results, they were accepted with the knowledge or intuition that they should not be uncritically applied. However, as this intuition and general understanding of mathematics shifted towards the concept based paradigm, exceptions became oddities — and counter examples became very powerful tools of argument in this new paradigm.

21.3.2 Counter examples and concepts

In the concept based paradigm, counter examples acquired a position much closer to their modern usage. As noted, counter examples are very instrumental in pointing out the differences between concepts and thereby helping to determine the extension of concepts. Used as tools of criticism, a theorem to which a counter example could be presented was certainly *false* in the concept based approach. There was no room for theorems with exceptions. In a sense, the concept based approach adhered to a viewpoint similar to the Lakatosian one that theorems with counter examples should either be discarded or modified to range over a

²²Strictly speaking, the function should also be defined at the origin, $f(0) = 0$. For a good discussion on this issue, see (Bottazzini 1990, lxix).

²³See section ??.

²⁴(Cauchy 1822, 277) and (Cauchy 1829, 394–395).

²⁵(Volkert 1986, 144–145).

smaller domain. There is an abundance of such applications of counter examples in the 1820s. ABEL presented one very elaborate example in his refutation of OLIVIER when he showed that *no* criterion of the proposed form could ever be constructed having the properties which OLIVIER had sought. However, the young mathematician who made the most use of counter examples in the 1820s and 1830s was probably DIRICHLET.

In 1829,²⁶ when DIRICHLET presented his famous result on the convergence of Fourier series, he started the paper with a scrutiny of an earlier paper by CAUCHY.²⁷ In particular, DIRICHLET criticized a point in the proof where CAUCHY had used an implicit assumption which DIRICHLET identified as follows: If the series $\sum a_n$ was convergent, any other series $\sum b_n$ such that $\lim \frac{b_n}{a_n} = 1$ would also be convergent. Against this argument, DIRICHLET presented the counter example

$$a_n = \frac{(-1)^n}{\sqrt{n}} \text{ and } b_n = \frac{(-1)^n}{\sqrt{n}} \left(1 + \frac{(-1)^n}{\sqrt{n}} \right)$$

of which the series $\sum a_n$ was convergent but the series $\sum b_n$ diverged. DIRICHLET described CAUCHY's conclusion as "not permissible" because it was easy to construct a counter example.²⁸

To DIRICHLET, the existence of one single, local counter example thus seems to have rendered the theorem false; in particular, we find none of the above remarks that "infinitely many similar counter examples may be found or constructed" in DIRICHLET's papers.²⁹ In some instances, a counter example led DIRICHLET to dismiss the faulty theorems as *false* and begin his own deductions from other principles. In other situations, DIRICHLET drew inspiration from his counter examples to revise existing proofs in ways which later led to proof analysis.

Later in the 19th century, counter examples acquired their modern status as complete refutations of theorems. To a mathematician educated at one of the German universities in the second half of the 19th century, a theorem could absolutely not admit exceptions and the precise formulation of theorems and proofs had truly become one of the trademarks of mathematics.

ABEL's use of counter examples seems to fall in both paradigms. As noted, sense can be made of ABEL's exception to *Cauchy's theorem* if it is interpreted in the formula based paradigm. On the other hand, ABEL's dismissal of OLIVIER's criterion of convergence shows signs of a concept based refutation. There, a single counter example was invoked to refute OLIVIER's claim and an elaborate analysis was employed to show that the concept of *tests of convergence* could not contain a

²⁶(Dirichlet 1829, 120).

²⁷(Cauchy 1827). Actually, DIRICHLET referred to a paper published in 1823 in the *Mémoires de l'Académie des Sciences*; but no paper with these details can be found in CAUCHY's *Œuvres*. Thus, it is here assumed that DIRICHLET actually meant (Cauchy 1827).

²⁸"*Mais cette conclusion n'est pas permise*" (Dirichlet 1829, 158).

²⁹(Dirichlet 1829; Dirichlet 1837).

criterion of a slightly generalized form.³⁰

The role of mathematical intuition. Importantly, the changing status of counter examples also reflects a change in a mental entity which may be called *mathematical intuition*.³¹ This intuition comprises the expertise, prejudices, and expectations of active mathematicians who have gained an insight into their objects and is thus part of KUHN's disciplinary matrix.

During the 18th century, mathematicians built up a high degree of insight into representations of functions, in particular into power series or other algebraic expressions. When this insight was formulated, it often took the form of formulae relating certain entities by means of algebraic notation and the formulae were considered to have aesthetic properties described as simplicity or degrees of symmetry. As illustrative examples, consider the solution formulae for general equations of low degree or the power series expansions of elementary transcendental functions. The art of mathematics also consisted of the trained ability to recognize patterns and manipulate representations to obtain various generalizations.

As a result of the change of paradigms, the contents and role of mathematical intuition also changed. A new kind of intuition emerged which helped mathematicians see differences and similarities between concepts and suggested ways of obtaining relations among concepts. As an indication of this change in intuition, mathematicians occasionally brought over intuitions from the old paradigm into the new one. This could lead them to generalize results into forms in which they were then no longer permitted. Thus, the changing intuitions are intimately connected with the process of *concept stretching* which LAKATOS has discussed as part of interpreting mathematical development.³²

21.4 Conclusion

The analytical scheme of a transition from a formula based paradigm to one based on concepts has shown its applicability in interpreting events in the disciplines of algebra and analysis in the 1820s. In particular, the role of new definitions, the coexistence of theorems and exceptions, and the new problems of delineation have contributed to throwing ABEL's mathematical production into perspective.

It is beyond the present scope to analyze and speculate as to the causal reasons for the purported change of paradigms in analysis and algebra. Neither is it the present purpose to discuss at length the general applicability of this frame of interpretation. However, it must be noticed that the interpretation might be

³⁰See chapter ??.

³¹For a discussion of mathematical intuition, see also (Volkert 1986).

³²E.g. (Lakatos 1976).

limited to the disciplines described here; in particular, it does not appear to be immediately applicable to geometry. With some right, one could argue that the transition could be interpreted simply as a maturing of the involved theories. Still, I believe that the simultaneous instances of the change of style as described above are sufficient to suggest that a general change in the modes of thought was involved.

New questions, new standards, new objects. In the presentation and analysis of ABEL's mathematical production, three local themes were introduced. Based on the description of his works in algebra, I have argued that a new type of questions was being introduced into mathematics. These new questions were indicative of the fundamental change of paradigms. Concerning ABEL's works in the foundations of analysis, it was illustrated how the change of basic definitions and standards of proof also reflected the new focus on concepts. Finally, a cross section of ABEL's works on new transcendentals illustrated how these transcendental objects were being treated with the help of algebraic methods and also how the introduction of new objects led to important questions of representation.

Throughout the description and analysis of ABEL's works, much attention has been paid to their mathematical contexts. The inspirations which ABEL drew from his predecessors and contemporaries have been described in order to illustrate how ABEL's works grew continuously out of the mathematical contexts. At the same time, ABEL's works were — at a number of points — remarkably novel and due attention has been paid to these aspects. To generalize, ABEL's methods and the problems which he attacked were generally well established whereas the questions which he raised and the approaches which he took in attacking these problems were often new and ground-breaking. In connection with the fundamental transition, this manifested itself in the sense that ABEL had one foot firmly placed in each of the two paradigms.

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