Using authentic sources in teaching logistic growth: A narrative design perspective

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History of mathematics for the benefit of future teachers Since the 1980s, history of mathematics has had to be an integrated part of the mathematics curriculum in the Danish upper-secondary schools. The arguments for teaching history of mathematics have spanned its potential for showing mathematics as a *human* and *creative* activity embedded in broader culture and responding to relevant questions and not as a fossilised or sterile discipline of formal reasoning and rote learning (Jankvist, 2009; Kjeldsen and Carter, 2014).

University courses in history of mathematics With such aims in mind, the university courses in history (and philosophy) of mathematics offer opportunities for both becoming acquianted with the context and contents of important mathematical concepts and episodes, initiating methodological reflections, and stimulating identity-formation in the students, whether they are future teachers or researchers. For the history of mathematics aimed at future teachers, we have identified four focal points (in addition to the learning objectives) that will aid the teacher in using history relevantly in mathematics teaching: tools for critical reading of i) primary and ii) secondary sources; and iii) historiographical reflections, in particular concerning iv) biographies of mathematicians.

Putting theory into practice In order to implement these ambitions, two challenges therefore arise for us: How do we (best) educate the teachers and how do we devise suitable material for classroom usage? Historians and educators of mathematics have addressed these two questions for decades (see e.g. Fried, 2001; Jahnke, 2002; Pengelley, 2011), yet in this presentation we present a new perspective on the design of teaching material using authentic sources and based on a situating narrative and a historical contextualisation.

Logistic growth through an authentic source Based in our teaching of a university course in the history of mathematics aimed at future teachers, we have developed teaching material about logistic growth. Central to the material is a short authentic mathematical source, namely Pierre-François Verhulst's article from 1838 in which he introduced what is now called the "logistic curve" (Verhulst, 1838; translated into English in Vogel et al., 1975). Although short, that source invites discussions of four important aspects of the use of history in upper-secondary mathematics education: teaching a central mathematical topic, illustrating and discussing the mathematical modeling process, addressing issues of philosophy of mathematics, and informing the students about the historical complexity of mathematics. Our material (Danielsen and Sørensen, 2014) includes a translation of the main source as well as a structured narrative providing the students with a window into the historical context and an understanding of Verhulst's thought process.

Narrative design perspective The central position of the authentic sources has allowed (and forced) us to situate it for the teachers and their students in a historical narrative that briefly introduces the main protagonists, their social and political contexts, and the mathematics of the early nineteenth century. Such a narrative approach is employed also to aid student identification with the problem at hand and make accessible the methods of solving it (see also Allchin, 2012).

Teaching core mathematics through a historical source In the Danish upper-secondary mathematics curriculum, differential equations and the logistic model are core topics. We think that, ideally, successful historical material should be chosen among the core topics or in close relation to them. Such choices make it possible to simultaneously teach mathematics and history of mathematics. And although many topics are not as prone to historical contextualization, a fair number of good historical sources exist that can be used in teaching elementary geometry, trigonometry, function theory, proofs, etc.

Discussions about mathematical models Mathematical models and the ability to critically assess them are key topics in mathematics classrooms. By using historical sources such as Verhulst's paper, the authentic modelling process with its incompleteness and contingent choices become visible to the students. In particular, Verhulst's aesthetic and pragmatic choices in building the model are otherwise difficult to illustrate. Thus, students get to "look into the workshop" of the mathematical modelling process, and what they see may nuance the picture often presented in textbooks of a finished, fossilised model. The fact that historical sources can help teachers identify *commognitive conflicts* has also been noticed in e.g. (Kjeldsen and Petersen, 2014).

Illustrating historical complexity and context The material is designed to include sufficient historical context to aid the teacher in situating the source. And the source is given in translation quite close to the original in order to allow the teacher to address historical complexities such as differences in notation or conceptions about mathematical notions and objects. These are, we believe, very important aspects that history can bring to the mathematics classroom.

Integration with other subjects In the Danish upper-secondary school system, collaboration between subjects is very important, both in the day-to-day teaching and in special projects. Thus, it becomes a special challenge to provide material suitable for the collaboration between mathematics and, for instance, history. Exercises 6 and 7, below, are examples of such collaborations.

Experiences from using material The material has been tested in different classes in different modules. In the design, it was important to allow for flexibility in the scope: the historical perspective could be used to briefly introduce logistic growth, or used throughout as a means of teaching logistic growth, or as the nexus for various collaborations with other subjects. These experiences have all been positive, and we are eagerly awaiting further feedback, in particular concerning the large student project in history and mathematics.

A template for further development In the mean time, we think that we have found a good and viable model for such use of authentic sources that can be extended to other cases. More materials are being developed, and ideally the format could be extended also beyond the present author collective.

References

Allchin, D. (2012). "The Minnesota Case Study Collection: New Historical Inquiry Case Studies for Nature of Science Education". *Science & Education*, vol. 21, pp. 1263–1281.

Danielsen, K. and H. K. Sørensen (2014). *Vækst i nationens tjeneste. Hvordan Verhulst fik beskrevet logistisk vækst.* København: Matematiklærerforeningen.

Fried, M. N. (2001). "Can Mathematics Education and History of Mathematics Coexist?" *Science & Education*, vol. 10, pp. 391–408.

Jahnke, H. N. et al. (2002). "The use of original sources in the mathematics classroom". In: *History in Mathematics Education. The ICMI Study.* Ed. by J. Fauvel and J. V. Maanen. New ICMI Study Series 6. Springer. Chap. 9, pp. 291–328.

Jankvist, U. T. (2009). "A categorization of the "whys" and "hows" of using history in mathematics education". *Educational Studies in Mathematics*, vol. 71, pp. 235–261. DOI: 10.1007/s10649-008-9174-9.

Kjeldsen, T. H. and J. Carter (2014). "The Role of History and Philosophy in University Mathematics". In: *International Handbook of Research in History, Philosophy and Science Teaching.* Ed. by M. R. Matthews. Dordrecht: Springer. Chap. 26, pp. 837–871.

Kjeldsen, T. H. and P. H. Petersen (2014). "Bridging History of the Concept of Function with Learning of Mathematics: Students' Meta-Discursive Rules, Concept Formation and Historical Awareness". *Science & Education*, vol. 23, no. 1, pp. 29–45. DOI: 10.1007/s11191-013-9641-2.

Pengelley, D. (2011). "Teaching with Primary Historical Sources: Should It Go Mainstream? Can It?" In: *Recent Developments on Introducing a Historical Dimension in Mathematics Education.* Ed. by V. Katz and C. Tzanakis. MAA Notes 78. Washington DC: Mathematical Association of America. Chap. 1, pp. 1–8.

Verhulst, P.-F. (1838). "Notice sur la loi que la population suit dans son accroissement". *Correspondance mathématique et physique de l'Observatoire de Bruxelles*, vol. 10, pp. 113–121.

— (1845). "Recherches mathématiques sur la loi d'accroissement de la population". *Nouveaux mémoires de l'Académie Royale des Sciences et Belles-Lettres de Bruxelles*, vol. 18, pp. 1–38.

Vogel, M. et al. (1975). "P. F. Verhulst's 'Notice sur la loi que la populations suit dans son accroissement' from Correspondence Mathematique et Physique. Ghent, Vol. X, 1838". *Physical Biological Sciences*, vol. 3, pp. 183–192.

Part 1 (§1–4 in Verhulst, 1838, pp. 113–114)

Note on the law that a population follows in its growth

Correspondance mathématique et physique de l'Observatoire de Bruxelles, vol. 10, 1838, p. 113–121.

It is known that the celebrated Malthus established as a principle that the human population tends to increase according to a geometric progression, so as to double after a certain period of time, for example, every twenty-five years. This proposition is incontestable, if one neglects the ever increasing difficulty of procuring supplies when the population has acquired a certain degree of agglomeration, or the resources which the population uses in its growth, even when the society is still nascent, such as a greater division of labor, the existence of a regular government and means of defense which insure the public tranquility, etc.

In effect, all things being equal in other respects, if a thousand souls have become two thousand after twenty-five years, these two thousand will become four thousand after the same lapse of time.

In our old European societies, where the fertile land has been cultivated for a long time, the work needed to improve a field already under cultivation can add little to its productivity; admitting that, in the first twenty-five years, the yield of the soil is doubled, in the second period, one will barely succeed in making it produce perhaps a third more. The virtual increase of the population thus finds a limit in the extent and fertility of the country, and the population, as a consequence, tends more and more to become stationary.

This is not always the case, because exceptions do exist; for example, when a civilized people cultivates a fertile territory, previously uninhabited, or when it has an industry which can provide great temporary benefits. In such instances, a large family becomes an asset, and the second generation finds it easier to establish itself than the first, because unlike the first, it does not have to struggle against the obstacles that the virgin state of the land presented to the first settlers.

Exercise 1 Reading this part of the source, consider these questions both from the perspective of the teacher and that of a student:

1. The name of the author is Pierre-François Verhulst (1804–1849). What can you find out about him on the internet?
2. How is Thomas Robert Malthus (1766–1834) described in the source? What can you find out about him online?
3. What can you find out about the context in which the source was produced?
4. How is the influence of agriculture and cultivation on populations described?

Part 2 (§5–9 in ibid., pp. 114–115)

In order to determine the rate at which the population grows in a given country, it is necessary to divide the increase of the population of each year by the population which produced it. This ratio, being independent of the absolute size of the population, can be regarded as the measure of this rate. If it is constant, the population grows in a geometric progression; if it is increasing, the progression is more than geometric, and of course less than geometric if it is decreasing.

Various hypotheses can be made on the retarding or sum of obstacles opposing the indefinite development of the population. Mr. Quetelet supposes it proportional to the square of the rate at which the population tends to grow.[a]

It is similar to the motion of a moving body which falls, through a viscous medium. The results of this comparison concur in a satisfactory manner with the statistical data, and with those which I obtained by my own formulas, when an indefinitely increasing density is attributed to the strata of the medium traversed.

The growth of the population necessarily has a limit, if only in the extent of land indispensible for the housing of that population. When a nation has consumed all the fruits of its fields, it can, it is true, procure provisions from other nations by the exchange of its products, and thus support a new increase in population. But it is evident that these imports must have limits, and must be checked, even long before the entire surface of the country is converted into towns. All the formulas by which one may attempt to represent the law of population must therefore satisfy the condition of allowing for a maximum which will only be reached in an extremely remote era. This maximum will be the total of the population which tends to become stationary.

I have tried for a long time to determine by analysis a probable law of population; but I have abandoned this type of research because the available data is too limited to allow the verification of my formula, so as to have no question about its accuracy. However, the course which I followed seemed to lead me to the understanding of the actual law, which when sufficient data becomes available, will support my speculations. Therefore, I thought it proper to aquiesce to the invitation of Mr. Quetelet and to present them in this article.

[a] *Essai de physique sociale*, 1st ed., p. 277.

Exercise 2 Reading this part of the source, consider these questions both from the perspective of the teacher and that of a student:

5. How is exponential growth described?
6. Population growth is compared to an example from physics. How?
7. Why do models of population growth require the inclusion of an upper limit?
8. Which problems have Verhulst had to overcome to build his model?

Part 3 (§10–11 in Verhulst, 1838, p. 115)

If p is the population, then dp is an infinitesimally small increase that it receives in a very short period of time dt. If the population increases by geometric progression, we would have the equation $\frac{dp}{dt} = mp$. However, as the rate of population growth is slowed by the very increase in the number of inhabitants, we must subtract from mp an unknown of p, so that the formula to be integrated can be written as

$$\frac{dp}{dt} = mp - \phi(p)$$

The simplest hypothesis that can be made on the form of the function ϕ is to suppose that $\phi(p) = np^2$. Then the integral of the above equation is found to be

$$t = \frac{1}{m}[log.\, p - log.\, (m - np)] + Constant$$

and three observations will be sufficient to determine the two coefficient constants m and n and the arbitrary constant.

By solving the last equation for p, it becomes

$$p = \frac{mp'e^{mt}}{np'e^{mt} + m - np'} \tag{1}$$

where p' designates the population at the boundary condition, $t = 0$, and e is the base of the napierian logrithms. If we put $t = \infty$, we see that the corresponding value of p is $P = \frac{m}{n}$. This is, therefore, the *upper limit of the population.*

Exercise 3 Reading this part of the source, consider these questions both from the perspective of the teacher and that of a student:

9. Which equation is eventually used to describe population growth?
10. Can you rework the equation into a form that is recognizable and understandable from a modern, school perspective?
11. How is the solution formulated in the source?
12. Use a CAS tool to find a solution. What do you find?
13. Can you find a relevant formula in your textbook? What is then the solution? Compare the various solutions obtained.
14. Which limit is contained in the formulas?

Part 4 (§12–16 in ibid., p. 116)

I tried successively: $\phi p = np^2$, $\phi p = np^3$, $\phi p = np^4$, $\phi p = n\, log.\, p$, and the differences between the populations calculated and those furnished by observation were approximately the same.

When the population increases more rapidly than geometric progression, the term $-\phi p$ becomes $+\phi p$; the differential equation can be integrated as in the preceding cases, but it must be understood that there can no longer be a maximum population.

I calculated some tables which follow from equation (1). The figures for France, Belgium and Essex county were drawn from official documents. Those concerning Russia are found in the work of Dr. Sadler, Law of Population, and I cannot guarantee their authenticity, not knowing by what method they were obtained. I could have extended the tables for France and Belgium up until 1837 by means of the Annuals published in these two countries since 1833, and thus verified my formula; but my other commitments did not give me sufficient time. This work was finished in 1833, and I have not touched it since.

I will remark in passing that the table which concerns France seems to show that the formula is very accurate, as the observations deal with larger numbers and were carried out more carefully. Nevertheless, the future alone will be able to reveal to us the true method of operation of the retarding force which we have represented by ϕp.

Exercise 4 Reading this part of the source, consider these questions both from the perspective of the teacher and that of a student:

15. Is the description of population growth unique?

16. In Verhulst' paper, he introduced four sets of data for populations in France, Belgium, Essex and Russia. These data can be downloaded from `www.matematikhistorie.dk/logistisk-vaekst`. How would you treat the data that Verhulst presented?

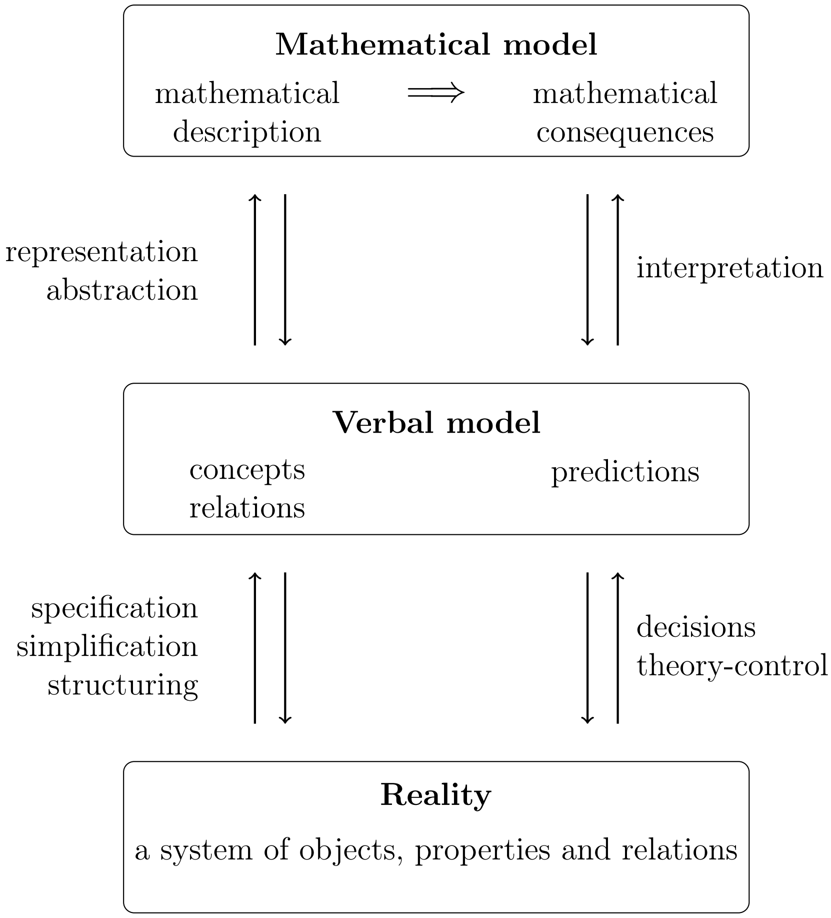

Figure 1: A simple model of mathematical modelling.

Exercise 5 In the source, Verhulst provided some remarks on the process of building his model and on the factors that influenced it. In this exercise, we put these into perspective and discuss the modelling process.

1. First outline the standard model of mathematical modelling as presented in figure 1. Be sure to explain how the mathematical model is related to reality by focusing on how the model is built and what kinds of statements about reality that it warrants.
2. Now analyse Verhulst's source to find those places in it where he explains or comments upon his model. What does he, for instance, say about the relation between the model and the actual population sizes? Again, you should focus on the construction of the model and the types of warrant that it provides.
3. Discuss the perspective on the process of building mathematical models which you have achieved in the first two tasks by discussing the types of knowledge that mathematical models can provide of the real world.

Exercise 6 Any mathematical creation is formed in a specific context, and it is often very important to know about this context in order to understand and explain the mathematics created, its purpose and its relevance. This exercise explores some relevant aspects surrounding Verhulst's modelling.

1. Start by comparing two maps of Europe from, say, 1800 and 1900 and describe the changes in nations and their borders that you see.
2. Search the internet for information on the so-called "Congress of Vienna" of 1814–1815 and describe in more details the changes to the nations of Denmark and The Netherlands.
3. Several European nations introduced new constitutions in the decades between 1830 and 1850: Belgium in 1831 and Denmark in 1848. The Danish constitution was partly modelled on the Belgium one. Analyse the constitutional rights of individual citizens granted by the Danish constitution of 1848 (chapter IV). Who were granted such rights? What shift of democratic power did this entail?
4. The new states were to a larger extend formed by (emerging) *criteria of national belonging*. Describe differences between the absolutist state and the nation state by focusing on the demarcation of the state.

The many new European states had a need for forming their own identity. This was partly a conscious process and partly the background for the emergence of new states in the first place.

5. Analyse some Danish national romantic works (poems and paintings) from the period 1830–1850, focusing on their role in forming a national identity. You should include knowledge from other subjects (Danish and history) in analysing the context and interpretation of these works.

An important part of the formation of new states consisted in the institutional development of a bureaucratic system for supporting parliamentarian and democratic decision processes.

6. The Danish *Statistical Bureau* was formed in 1849 (notice the year) as the precursor of the modern *Danmarks Statistik*. What can you find out about this bureau and the background for its establishment?
7. The geographic survey of Denmark was begun under the auspices of the Royal Danish Academy for Sciences towards the end of the eighteenth century. What can you find out about this project and the mathematicians who were involved in it?

Exercise 7 A central element of Verhulst's modelling is the assumption that social and human relations are regulated by *laws* and can be studied in the same way as (natural) science studies Nature. This assumption and its consequences are treated in this exercise.

1. The conception of a "social physics" as the study of social relations subjected to laws analogous to the laws of nature goes back to the scientists and philosophers Adolphe Quetelet (1796–1874) and Auguste Comte (1798–1857). What does Quetetelet mean when he in *Sur l'homme et le développement de ses facultes, ou essai de physique sociale* (1835) talks about "the average man"? What else can you find out about the social physics of Quetelet and Comte?

Human beings and their properties have been quantified in many different ways over time. One of the purposes of associating numbers to individuals has been to define what it means to be 'normal' — and sometimes 'ideal'.

2. In the arts (paintings and in particular sculptures), human proportions have been associated with numbers in order to describe 'the ideal body'. Analyse selected works by e.g. Vitrivius (ca. 75 BC–ca. 25 BC), Leonardo Da Vinci (1452–1519) or Corbusier (1887–1965), focusing on their description and use of ideal human proportions.

3. Attempts have also been made to use quantify non-physical aspects of human beings like personality, intelligence or crime. This became most explicit in the pseudo-science *phrenology* which was taken quite seriously in the nineteenth century. What can you find out about phrenology?

4. In order to illustrate how quantificaton can be used to define *normality*, do some research on the notion of 'body mass index' which was first suggested by Quetelet.

This far, you have explored some applications of quantification of social and human domains. These are not value-neutral as you will now explore and analyse.

5. Present Malthus' political and philosophical position which is the foundation for his analysis of population growth. Which views of revolution and democracy are behind his position?

6. The quantification of individuals are of course an important part of democratic government in which all votes are equal (with some provision for representative democracy). But it is not (and has not been) unproblematic to decide *which* votes and voices are to be counted. Analyse selected democracy-critical thinkers with special attention given to their presentation of the quantification of individuals.

7. The quantification of individuals can also lead to *alienation* and the loss of individuality and identity. Analyse selected works (fiction and movies) for their presentation fo this form of alienation through quantification. Examples could include Dickens' novel *Hard Times* (1854) or episode of the TV-series "The Prisoner" (1967). Sometimes, such alienations are related to certain political doctrines, so discussions should also include the previous task.

Year	Population
1790	3.929.827
1800	5.305.925
1810	7.239.814
1820	9.638.151
1830	12.866.020
1840	17.062.566

(a) Tables of US population, 1790–1840 from Verhulst (1845).

Year	Population	r
1790	3.930.000	
1795	4.618.000	
1800	5.306.000	
1805	6.273.000	
1810	7.240.000	
1815	8.439.000	2,147
1820	9.638.000	2,087
1825	11.252.000	2,120
1830	12.866.000	2,052
1835	14.964.000	2,076
1840	17.063.000	2,021

(b) Table of computed values based on Verhulst (1845).

Table 1: Verhulst's tables showing the population of the United States.

Exercise 8 This exercise examines Verhulst's predictions for the population of the United States based on the data at his disposal in 1845.

a) Determine the mathematical expression for population growth in table 1(a) by means of exponential regression. Determine T_2 (the time of duplication).

Based on table 1(a), Verhulst computes the table 1(b).

b) In table 1(b), what has happened to the populations for the years 1790, 1800, etc?

c) How are the populations for the years 1795, 1805, 1815, 1825 and 1835 determined? How does that fit with Verhulst's stated assumption that the population grows exponentially?

d) How has Verhulst found his r? How would we interpret his r? (Hint: Verhulst wants to show that T_2 is 25 years). Notice that Verhulst has made some errors of rounding in some of his calculations of r.

e) Compare Verhulst's results with your own obtained in task a). Think about the differences in the way Verhulst thought about this problem compared to how we would solve it today. Which challenges have Verhulst had to face with his approach?

Exercise 9 In the source, Verhulst *chose* to consider the *correction term* $\phi(p) = np^2$ as a way of improving on Malthus' claim of exponential growth. In this exercise, we explore an alternative to this particular model.

1. Assume $\phi(p) = np^3$ for a constant n. First formulate the differential equation parallel to Verhulst's one in §10 for this choice of ϕ.
2. Now, derive an expression for t as a function of p (and the constants m and n).
3. Next, from the derived expression, develop a method to determine the parameters m and n from three sets of data distributed with equal time spans. That is, from the values $p(t - \Delta t), p(t), p(t + \Delta t)$ you should find a way of determining m, n by a formula.
4. From the formula that you found in task 2, derive a new formula for p as a function of t (and the parameters m and n). This formula should thus be analogous to the one obtained by Verhulst in §11.
5. What happens to the $p(t)$ that you found in the previous task when t becomes very large ($t \to \infty$)? For this task, you can use *l'Hospital's rule* but you can also do without. Find an expression for $\lim_{t\to\infty} p(t)$ as a function of the parameters m, n.
6. Now consider Verhulst's data set for the poulation of Belgium and focus on the years 1815, 1824 og 1833. Determine the parameters m and n using the formula that you found in task 3.
7. Use the formula from task 5 to determine the limit population of Belgium based on the parameters found in task 6.
8. Use a CAS program to plot data for the population of Belgium together with the function found in task 4 and the function found using Verhulst's model (§11). Discuss whether Verhulst is correct in his statement that both models fit the data equally well.
9. Compare your value for the limit population of Belgium found in task 7 to the limit population predicted by Verhulst's model.
10. Verhulst's table with data for the population of Belgium has only a small number of observations over a short period of time. Use data for the population of Denmark from 1900 to 2000 to repeat the investigations from tasks 8 and 9. Discuss whether Verhulst's claim can be said to be supported by your observations.