Isaac Newtons bevis for Herons formel (1728)

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I „Herons formel“ (Danielsen og Sørensen, 2016) gengives og diskuteres det bevis for Herons formel, som Isaac Newton (1643–1727) gav.

Newton formulerede problemet således:

> Havende givet siderne og grundlinjen i en vilkårlig retlinjet trekant, at finde segmenterne af grundlinjen, den vinkelrette, arealet, og vinklerne.

Beviset forløber hos Newton således:

> Betragt trekanten ABC og lad siderne være AC og BC og grundlinjen AB. Halvér AB i I og afsæt på den (forlænget i begge retninger) AF og AE lig med AC samt BG og BH lig med BC [...]. Nedfæld den vinkelrette CD fra C på grundlinjen. Og så vil $ACq - BCq$ være $= ADq + CDq - CDq - BDq = ADq - BDq = \overline{AD + BD} \times \overline{AD - BD} = AB \times 2DI$. Derfor $\frac{ACq - BCq}{2AB} = DI$. [...]
>
> Fra IE, dvs. fra $AC - \frac{1}{2}AB$, trækkes DI, og der bliver tilbage $DE = \frac{BCq - ACq + 2AC \times AB - ABq}{2AB}$, dvs. $= \frac{\overline{BC + AC - AB} \times \overline{BC - AC + AB}}{2AB}$, eller $= \frac{HE \times EG}{2AB}$. Træk DE fra FE, eller $2AC$, og der bliver tilbage $FD = \frac{ACq + 2AC \times AB + ABq - BCq}{2AB}$, dvs. $= \frac{\overline{AC + AB + BC} \times \overline{AC + AB - BC}}{2AB}$, eller $= \frac{FG \times FH}{2AB}$. Og eftersom CD er en mellemproportional mellem DE og DF, [...] så vil CD være $= \frac{\sqrt{FG \times FH \times HE \times EG}}{2AB}$ [...]. Multiplicér CD med $\frac{1}{2}AB$, og du vil have arealet $= \frac{1}{4}\sqrt{FG \times FH \times HE \times EG}$. [...]

REFERENCER

Danielsen, Kristian og Henrik Kragh Sørensen (apr. 2016). „Herons formel. Hvordan en aleksandriner fik sat mål på alle slags trekanter“. Kildecentreret matematikhistorie på STX. Accepteret.

Newton, Isaac (1967). *The Mathematical Works of Isaac Newton*. Udgivet og med en indledning skrevet af Derek T. Whiteside. 2 bd. The Sources of Science. New York og London: Johnson Reprint Corporation.

to $\frac{1}{2}$ CE or CE to DE) so Radius to the Sine of $\frac{1}{2}$ the Angle A.

VI. And as a mean Proportional between $2a$ and $2b$ to a mean Proportional between $a+b+c$ and $a+b-c$ (so CE to CD) so Radius to the Cosine of half the Angle A.

But if besides the Angles, the Area of the Triangle be also sought, multiply CDq by $\frac{1}{4}$ ABq, and the Root, *viz.* $\frac{1}{4}\sqrt{\overline{a+b+c}\times\overline{a+b-c}\times\overline{a-b+c}\times\overline{-a+b+c}}$ will be the Area sought.

Problem XII.

Having the Sides and Base of any right lined Triangle given, to find the Segments of the Base, the Perpendicular, the Area, and the Angles. [See Figure 40.]

Let there be given the Sides AC, BC, and the Base AB of the Triangle ABC. Bisect AB in I, and take on it (being produced on both Sides) AF and AE equal to AC, and BG and BH equal to BC. Join CE, CF; and from C to the Base let fall the Perpendicular CD. And $ACq-BCq$ will be $= ADq+CDq-CDq-BDq = ADq-BDq = \overline{AD+BD}\times\overline{AD-BD} = AB\times 2DI$. Therefore $\frac{ACq-BCq}{2AB} = DI$. And $2AB : AC+BC :: AC-BC : DI$. Which is a Theorem for determining the Segments of the Base.

From IE, that is, from $AC-\frac{1}{2}AB$, take away DI, and there will remain $DE = \frac{BCq-ACq+2AC\times AB-ABq}{2AB}$, that is $= \frac{\overline{BC+AC-AB}\times\overline{BC-AC+AB}}{2AB}$, or $= \frac{HE\times EG}{2AB}$. Take away DE from FE, or 2AC, and there will remain $FD = \frac{ACq+2AC\times AB+ABq-BCq}{2AB}$; that is $= \frac{\overline{AC+AB+BC}\times\overline{AC+AB-BC}}{2AB}$, or $=$

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$\frac{FG\times FH}{2AB}$. And since CD is a mean Proportional between DE and DF, and CE a mean Proportional between DE and EF, and CF a mean Proportional between DF and EF, CD will be $= \frac{\sqrt{FG\times FH\times HE\times EG}}{2AB}$, $CE = \sqrt{\frac{AC\times HE\times EG}{AB}}$, and $CF = \sqrt{\frac{AC\times FG\times FH}{AB}}$. Multiply CD into $\frac{1}{2}$AB, and you will have the Area $= \frac{1}{4}\sqrt{FG\times FH\times HE\times EG}$. But for determining the Angle A, there come out several Theorems:

1. As $2AB\times AC : HE\times EG$ (:: AC : DE) :: Radius : versed Sine of the Angle A.
2. $2AB\times AC : FG\times FH$ (:: AC : FD) :: Radius : versed Cosine of A.
3. $2AB\times AC : \sqrt{FG\times FH\times HE\times EG}$ (:: AC : CD) :: Radius : Sine of A.
4. $\sqrt{FG\times FH} : \sqrt{HE\times EG}$ (:: CF : CE) :: Radius : Tangent of $\frac{1}{2}$ A.
5. $\sqrt{HE\times EG} : \sqrt{FG\times FH}$ (:: CE : FC) :: Radius : Cotangent of $\frac{1}{2}$ A.
6. $2\sqrt{AB\times AC} : \sqrt{HE\times EG}$ (:: FE : CE) :: Radius : Sine of $\frac{1}{2}$ A.
7. $2\sqrt{AB\times AC} : \sqrt{FG\times FH}$ (:: FE : FC) :: Radius : Cosine of $\frac{1}{2}$ A.

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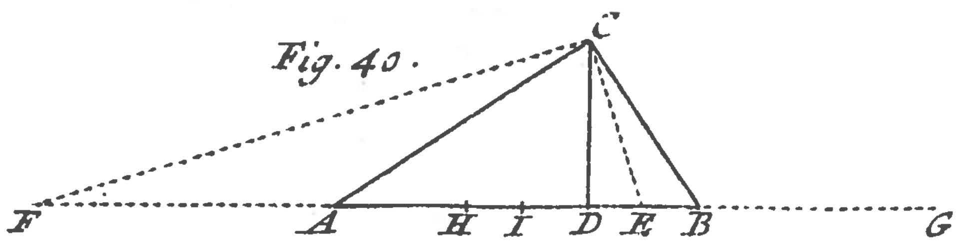

Figur 1: Newtons bevis fra 1728 med tilhørende figur; gengivet fra (Newton, 1967, bd. 2, s. 62).